Available online at www.sciencedirect.com
SCIENCE @ DIRECT•
ELSEVIER
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech
Journal of Applied Mathematics and Mechanics 68 (2004) 583–595

# SWITCHING SURFACES IN LINEAR DIFFERENTIAL GAMES WITH A FIXED INSTANT OF TERMINATION†

V. S. PATSKO
Ekaterinburg
e-mail: patsko@imm.uran.ru

(*Received* 17 *February* 2004)

Antagonistic linear differential games with a fixed instant of termination and a continuous terminal pay function are considered. The control action of the first (minimizing) player is assumed to be scalar and bounded in modulus. The vector control of the second player is restricted by a geometrical constraint. An assertion is proved concerning the sufficient condition and, when this is satisfied, the optimal negative feedback positional control of the first player can be specified using the switching surface which separates the space of the game into two parts, in each of which there is its own limit value of the control action. The proposed control procedure is stable with respect to inaccuracies in the numerical construction of the switching surface. 

## 1. FORMULATION OF THE PROBLEM AND THE MAIN RESULT

*Preliminary description of the problem.* Suppose a linear differential game with a fixed instant of termination $\vartheta$ is described by the relations

$$\begin{gathered}\dot{y}(t) = B^{(1)}(t)u(t) + C^{(1)}(t)\upsilon(t)\\ y(t)\in R^n,\quad |u(t)|\le\mu,\quad \upsilon(t)\in Q^{(1)};\quad \gamma^{(1)}(y(\vartheta))\end{gathered}\tag{1.1}$$

We stipulate that the control action $u(t)$ of the first player is scalar and bounded in modulus by the number $\mu > 0$. We assume that the set $Q^{(1)}$, which constraints the control action $\upsilon(t)$ of the second player, is convex compactum in a finite-dimensional space. Hence, $B^{(1)}(t)$ is a column-vector and $C^{(1)}(t)$ is a matrix of the corresponding dimensions. The functions $B^{(1)}$, $C^{(1)}$ are assumed to be piecewise-continuous. Suppose $\gamma^{(1)}: R^n \to R$ is a continuous pay function. The first player minimizes the value of $\gamma^{(1)}(y(\vartheta))$ while the interests of the second player are the opposite.

We will call the game (1.1) the initial game. The notation referring to it is given the superscript (1). We stipulate that the initial instants $t_0$ belong to the interval $T = [\vartheta_1, \vartheta]$, where $\vartheta_1 < \vartheta$. Suppose $Z = T\times R^n$ is the space of the game. We call a measurable function of the time $t\to u(t)$ $(t\to\upsilon(t))$, which satisfies the constraint $|u(t)|\le\mu$ $(\upsilon(t)\in Q^{(1)})$ for any $t$, a permissible preset control $u(\cdot)$ $(\upsilon(\cdot))$ of the first (second) player. We will denote the set of all permissible preset controls $\upsilon(\cdot)$ of the second player by $L^{(1)}$.

Following the well-known procedure [1], we will consider the arbitrary functions $(t, x)\to U(t, x)$, defined in the set $Z$ with numerical values which are bounded in modulus by the number $\mu$, as the permissible positional strategies of the first player. We will denote by the symbol $y^{(1)}(\cdot; t_0, x_0, U, \Delta, \upsilon(\cdot))$ the stepwise motion of system (1.1) from the position $(t_0, x_0)$, when the first player uses a strategy $U$ in a discrete control scheme [1] with a step size $\Delta > 0$, while a control $\upsilon(\cdot)\in L^{(1)}$ is a realized for the second player.

We put

$$\Gamma^{(1)}(t_0, x_0, U, \Delta) = \sup_{\upsilon(\cdot)\in L^{(1)}}\gamma^{(1)}(y^{(1)}(\vartheta; t_0, x_0, U, \Delta, \upsilon(\cdot)))$$

†*Prikl. Mat. Mekh.* Vol. 68, No. 4, pp. 653–666, 2004.

0021–8928/$—see front matter. 
doi: 10.1016/j.jappmathmech.2004.07.011

The quantity $\Gamma^{(1)}(t_0, x_0, U, \Delta)$ has the meaning of a guarantee which offers the first player a strategy $U$ for the initial position $(t_0, x_0)$ in the discrete control scheme with a step size $\Delta$. The best guarantee for the first player in the case of the initial position $(t_0, x_0)$ is defined by the formula

$$\Gamma^{(1)}(t_0, x_0) = \min_U \overline{\lim_{\Delta \to 0}} \Gamma^{(1)}(t_0, x_0, U, \Delta)$$

where $\overline{\lim}$ denotes an upper limit. It has been shown [1] that a minimum with respect to $U$ is reached, that is, an optimal strategy exists. At the same time, a dependence of the optimal strategy of the first player on the initial position $(t_0, x_0)$ is not ruled out.

It is well known [1, 2] that the best guaranteed result $\Gamma^{(1)}(t_0, x_0)$ is identical with the symmetrically determined best guaranteed result of the second player. The quantity $\Gamma^{(1)}(t_0, x_0)$ is therefore also called the value of the value function at the point $(t_0, x_0)$.

It will be shown below that, in the case of a certain additional condition in game (1.1), a universal, optimal strategy $U^*$ of the first player exists which is stable with respect to errors in its numerical specification.

Universality means that the strategy $U^*$ is optimal for all initial positions $(t_0, x_0) \in Z$. We stress that we are talking of universality in a "rigorous" sense: the strategies being considered are solely functions of the arguments $t$ and $x$. In the class of strategies which additionally depend on a certain "accuracy parameter", the existence of optimal, universal strategies has been established earlier [3] for an extensive class of problems.

The universal optimal strategy $(t, x) \to U^*(t, x)$ will be determined using a "switching surface" which divides up the space of the game $Z$ into two parts: on one side the control $u$ takes the value $-\mu$ and, on the other side, the value $+\mu$. In the switching surface itself, the optimal value of the control $u$ can take any value from the interval $[-\mu, \mu]$.

The question of the existence of universal optimal strategies in differential games has been concisely discussed [1, p. 48] and became sharper after the appearance of the paper [4] in which an example of a game problem was cited where an universal optimal strategy does not exist. It has been shown [5, 6] that a universal optimal strategy of the first player exists in the case of linear differential games of the form of (1.1) with a convex pay function and that it can be specified using the switching surface. The stability of this strategy was based [7] on an assumption concerning the boundedness of the "velocity of rotation" of the vector $B^{(1)}(t)$.

It has been established in [8, 9] that, if the set $Q^{(1)}$ is an interval (that is, the control action $v$ is scalar) then a universal optimal strategy of the second (maximizing) player exists and it can also be specified using the switching surface. However, this strategy does not possess the property of stability.

In this paper, the results obtained in [7] are reinforced: the condition of the convexity of the pay function is relaxed and the assumption concerning the boundedness of the "velocity of rotation" of the vector $B^{(1)}(t)$ is removed. As in [7], the following scheme of reasoning issued. Guided by computer syntheses, we replaced the initial differential game with a convenient approximating game for which we can construct a certain $u$-stable [1, 2] function or even the value function of the game. On processing this function, we obtain the switching surface. We now use this resulting switching surface in the initial differential game in order to specify the universal strategy of the first player. We estimate the guarantee of the first player which it ensures, using the universal strategy constructed. As a consequence, we obtain a result from this estimate which holds true for the universal, optimal, stable strategy in the game (1.1).

We now make a remark concerning the description of the dynamics of a linear differential game in the form of (1.1). A special feature of this description is the fact that the phase variable does not enter into the right-hand side. Suppose a linear differential game with a fixed instant of termination $\vartheta$ has the form

$$\dot{\mathbf{y}}(t) = \mathbf{A}(t)\mathbf{y}(t) + \mathbf{B}(t)u(t) + \mathbf{C}(t)v(t)$$
$$\mathbf{y}(t) \in R^m, \quad |u(t)| \le \mu, \quad v(t) \in Q^{(1)}; \quad \gamma(\mathbf{y}(\vartheta))$$

We assume that the pay function $\gamma$ is solely determined by the values of certain $n$ coordinates, $n \le m$, of the phase vector at the instant of termination. Then, the transition to the form (1.1) is achieved [1, p. 160] using a standard transformation $y(t) = X_{n,m}(\vartheta, t)\mathbf{y}(t)$, where $X_{n,m}(\vartheta, t)$ is an $n \times m$ matrix composed for the corresponding $n$ rows of the fundamental Cauchy matrix for the system $\dot{\mathbf{y}}(t) = \mathbf{A}(t)\mathbf{y}(t)$. In this case

$$B^{(1)}(t) = X_{n,m}(\vartheta, t)\mathbf{B}(t), \quad C^{(1)}(t) = X_{n,m}(\vartheta, t)\mathbf{C}(t), \quad \gamma^{(1)}(y(\vartheta)) = \gamma(\mathbf{y}(\vartheta))$$

*Approximating game.* Together with game (1.1), we consider a further differential game

$$\begin{aligned} \dot{y}(t) &= B^{(2)}(t)u(t) + C^{(2)}(t)\upsilon(t) \\ y(t) &\in R^n, \quad |u(t)| \le \mu, \quad \upsilon(t) \in Q^{(2)}; \quad \gamma^{(2)}(y(\vartheta)) \end{aligned} \tag{1.2}$$

with a fixed instant of termination $\vartheta$. We shall interpret game (1.2) as an approximation of game (1.1), which is convenient for computer calculations. Here $y(t)$ is the phase vector, and the functions $B^{(2)}$ and $C^{(2)}$ are piecewise-continuous. The constraint on the scalar control action of the first player is the same as in game (1.1) and the set $Q^{(2)}$ is a compactum in a finite-dimensional space. We assume that the continuous pay function $\gamma^{(2)} : R^n \to R$ satisfies the Lipschitz condition with a constant $\lambda$ and the condition $\gamma^{(2)}(x) \to \infty$ when $|x| \to \infty$. The first player minimizes the value of $\gamma^{(2)}(y(\vartheta))$ and the second player maximizes it.

Quantities belonging on the approximating game are indicated by the superscript (2). We determine the permissible preset controls $u(\cdot)$, $\upsilon(\cdot)$ of the first and second players in the same way as was done in the case of game (1.1). The set of all permissible preset controls $\upsilon(\cdot)$ of the second player is denoted by $L^{(2)}$.

We shall assume that a certain continuous $u$-stable function $V^{(2)} : Z \to R$ with the boundary condition

$$V^{(2)}(\vartheta, x) = \gamma^{(2)}(x), \quad x \in R^n$$

is constructed within the framework of the approximating game (1.2). According to the well-known definition [1, 2], we say that the function $V^{(2)}$ is $u$-stable if, for any position $(t_*, x_*) \in Z$, for any $t^* \in (t_*, \vartheta]$ and for any $\upsilon(\cdot) \in L^{(2)}$, a permissible preset control $u(\cdot)$ of the first player is found such that the inequality

$$V^{(2)}(t^*, y^{(2)}(t^*)) \le V^{(2)}(t_*, x_*)$$

is satisfied for the motion $y^{(2)}(t) = y^{(2)}(t; t_*, x_*, u(\cdot), \upsilon(\cdot))$.

We assume that the function $V^{(2)}$ satisfies the Lipschitz condition with a constant $\lambda$ with respect to the argument $x$ uniformly with respect to $t \in T$. If $V^{(2)}$ is the value function of the game (1.2), then the satisfaction of this property follows from the condition imposed on the function $\gamma^{(2)}$.

We introduce the function $B^{(3)} : T \to R^n$ which satisfies the Lipschitz condition with a constant $\beta$. Interestingly, $B^{(3)}$ can be treated as a Lipschitz approximation to the functions $B^{(1)}$ and $B^{(2)}$. We use the following notation

$$\sigma = \max_{t \in T} |B^{(3)}(t)|$$

The concept of quasiconvexity of a scalar function is used below. As usual, this means the convexity of its level sets (Lebesgue sets).

*Condition* A. For any $t \in T$ for which $B^{(3)}(t) \ne 0$, the contraction of function $V^{(2)}(t, \cdot)$ in any line in $R^n$ parallel to the vector $B^{(3)}(t)$ is a quasiconvex function.

*Remark.* We consider a function which is a contraction for the function $V^{(2)}(t, \cdot)$ in a certain line which is parallel to the vector $B^{(3)}(t)$. The condition which has been formulated implies the requirement of a non-rigorous monotonicity of this one-dimensional function along both sides of the point of its global minimum.

Condition A is satisfied, in particular, if the function $V^{(2)}(t, \cdot)$ is quasiconvex for any $t \in T$. In the case when $V^{(2)}$ is the value function of the approximating game (1.2), it is sufficient to require that the pay function $\gamma^{(2)}$ is quasiconvex in order to ensure the quasiconvexity of the functions $V^{(2)}(t, \cdot)$, $t \in T$.

*The switching surface. The multivalued function* $\mathrm{U}^0$. For $(t, x) \in Z$, we put

$$\mathcal{A}(t, x) = \{z \in R^n : z = x + \alpha B^{(3)}(t), \ \alpha \in R\}$$

If $B^{(3)}(t) \ne 0$, then the set $\mathcal{A}(t, x)$ is a straight line which passes in the space $R^n$ through a point $x$ parallel to the vector $B^{(3)}(t)$. In the case when $B^{(3)}(t) = 0$, the set $\mathcal{A}(t, x)$ is degenerate and coincides with the point $x$. Without picking out the degenerate case separately, we shall always call the set $\mathcal{A}(t, x)$ a straight line.

Suppose that

$$\mathcal{V}(t,x) = \min_{z \in \mathcal{A}(t,x)} V^{(2)}(t,z), \quad (t,x) \in Z$$

A minimum is attained since the function $V^{(2)}(t,\cdot)$ is continuous and parts to infinity when $|x| \to \infty$. By virtue of Condition A, the set of points of the minimum is an interval. If $B^{(3)}(t) = 0$, then $\mathcal{V}(t,x) = V^{(2)}(t,x), x \in R^n$.

Next, suppose that for all $t \in T$

$$\Pi(t) = \{x \in R^n : V^{(2)}(t,x) = \mathcal{V}(t,x)\}$$

$$\Pi_-(t) = \{x \in R^n : x + \alpha B^{(3)}(t) \notin \Pi(t), \forall \alpha \geq 0\}$$

$$\Pi_+(t) = \{x \in R^n : x + \alpha B^{(3)}(t) \notin \Pi(t), \forall \alpha \leq 0\}$$

The set $\Pi_-(t)$, $\Pi_+(t)$ are located in the space $R^n$ on different sides relative to the set $\Pi(t)$. It follows from condition A that, for any $(t,x) \in Z$, the function $V^{(2)}(t,\cdot)$ does not increase (does not decrease) in the direction of the vector $B^{(3)}(t)$ at the intersection of the straight line $\mathcal{A}(t,x)$ with the set $\Pi_+(t)$, ($\Pi_-(t)$).

We define the multivalued function

$$\mathbf{U}^0(t,x) = \begin{cases} \{-\mu\}, & x \in \Pi_-(t) \\ \{\mu\}, & x \in \Pi_+(t) \\ [-\mu,\mu], & x \in \Pi(t) \end{cases}$$

in $Z$.

The function $\mathbf{U}^0(t,\cdot)$ takes limiting values from the interval $[-\mu,\mu]$ in the sets $\Pi_-(t)$, $\Pi_+(t)$ and "switches" from one limiting value to the other in the set $\Pi(t)$. The set

$$\Pi = \{(t,x) \in Z : x \in \Pi(t)\}$$

is closed, simply connected set which subdivides $Z$ into two parts. Although the set $\Pi$ is not always a surface in the generally accepted sense, for clarity we shall nevertheless call it the switching surface of the control action of the first player.

*The set* $\Pi^r(t)$. *The multivalued function* $\mathbf{U}^r$. We continue to introduce the notation for formulating the basic result.

Suppose $r \geq 0$. In the case when $B^{(3)}(t) \neq 0$, we put

$$\Pi^r(t) = \left\{x \in R^n : x = z + \alpha\frac{B^{(3)}(t)}{|B^{(3)}(t)|}, \quad z \in \Pi(t), \quad |\alpha| \leq r\right\}$$

The set $\Pi^r(t)$ is a geometric $r$-expansion of the set $\Pi(t)$. The expansion occurs with the use of the vector $B^{(3)}(t)$. If $B^{(3)}(t) = 0$, we assume that $\Pi^r(t) = \Pi(t) = R^n$.

We introduce the sets

$$\Pi^r_-(t) = \{x \in R^n : x + \alpha B^{(3)}(t) \notin \Pi^r(t), \forall \alpha \geq 0\}$$

$$\Pi^r_+(t) = \{x \in R^n : x + \alpha B^{(3)}(t) \notin \Pi^r(t), \forall \alpha \leq 0\}$$

The set $\Pi^r_-(t)$ ($\Pi^r_+(t)$) is the part of the space $R^n$ which is located, relative to $\Pi^r(t)$, along the direction of (in the opposite direction to) the vector $B^{(3)}(t)$. It is obvious that $\Pi^r_-(t) \subset \Pi_-(t)$, $\Pi^r_+(t) \subset \Pi_+(t)$. When $r = 0$, we have $\Pi^r(t) = \Pi(t)$, $\Pi^r_-(t) = \Pi_-(t)$, $\Pi^r_+(t) = \Pi_+(t)$.

We define the multivalued function in $Z$

$$\mathbf{U}^r(t,x) = \begin{cases} \{-\mu\}, & x \in \Pi^r_-(t) \\ \{\mu\}, & x \in \Pi^r_+(t) \\ [-\mu,\mu], & x \in \Pi^r(t) \end{cases}$$

*Formulation of the basic result.* For any instants $t_*$ and $t^*$ from the interval $T$, we put

$$\chi(t_*, t^*) = \mu\int_{t_*}^{t^*}\kappa(t)dt + \int_{t_*}^{t^*}m(t)dt$$

$$\kappa(t) = |B^{(1)}(t) - B^{(3)}(t)| + |B^{(2)}(t) - B^{(3)}(t)|$$

$$m(t) = \max_{l \in R^n,\, |l| \le 1}\left[\max_{q \in Q^{(1)}} l'C^{(1)}(t)q - \max_{q \in Q^{(2)}} l'C^{(2)}(t)q\right]$$

The quantity $\chi(t_*, t^*)$ characterizes the difference in the functions $B^{(1)}$, $B^{(2)}$ and $B^{(3)}$ as well as the functions $C^{(1)}$ and $C^{(2)}$ and the sets $Q^{(1)}$ and $Q^{(2)}$ in an integral sense. A prime denotes transposition.

Assuming that the initial positions of system (1.1) belong to a certain compact set $K$ in the space of the game $Z$, we denote the compact set in $R^n$, which determines from above the set of possible states of system (1.1) at the instant $\vartheta$, by the symbol $F$. It is assumed that

$$\|\gamma^{(1)} - \gamma^{(2)}\|_F = \max_{x \in F}|\gamma^{(1)}(x) - \gamma^{(2)}(x)|$$

The following assertion will be proved next.

*Theorem.* Suppose the conditions, including Condition A, imposed on systems (1.1) and (1.2), and also on the functions $V^{(2)}$ and $B^{(3)}$, are satisfied. Suppose $r \ge 0$, $\Delta > 0$. Then, the estimate

$$\begin{aligned}\Gamma^{(1)}(t_0, x_0, U, \Delta) &\le V^{(2)}(t_0, x_0) + \Lambda(t_0, r, \Delta) + \|\gamma^{(1)} - \gamma^{(2)}\|_F \\ \Lambda(t_0, r, \Delta) &= 2\lambda\sqrt{(2\sigma\mu\Delta + r)\beta\mu}(\vartheta - t_0) + 4\lambda\sigma\mu\Delta + \lambda r + \lambda\chi(t_0, \vartheta)\end{aligned} \tag{1.3}$$

holds for any strategy $U$ of the first player such that $U(t, x) \in \mathbf{U}^r(t, x)$, $(t, x) \in Z$ and any initial position $(t_0, x_0) \in K$.

We will now give several explanations. The function $V^{(2)}$, which possesses the property of $u$-stability, is assumed to have been constructed within the framework of the approximating game. There is therefore a known value of $V^{(2)}(t_0, x_0)$ on the right-hand side of limit (1.3). The difference in the dynamics of the initial and the approximating games, as well as the difference in the function $B^{(3)}$ from the functions $B^{(1)}$ and $B^{(2)}$, are taken into account by the quantity $\chi(t_0, \vartheta)$. The term $\|\gamma^{(1)} - \gamma^{(2)}\|_F$ characterizes the difference in the pay functions. The switching sets $\Pi^r(t)$, $t \in T$ for the multivalued function $\mathbf{U}^r$ are defined in terms of constructions which are implemented using the functions $V^{(2)}$ and $B^{(3)}$.

On the whole, the right-hand side of relation (1.3) estimates the guarantee of the first player in the game (1.1) when it uses an arbitrary, single-valued positional strategy $U$, which is a sample from the multivalued function $\mathbf{U}^r$.

Since $\Pi(t) \subset \Pi^r(t)$, $t \in T$, then, outside the sets

$$\Pi^r = \{(t, x) \in Z : x \in \Pi^r(t)\}$$

the strategy $U$ is identical with the function $\mathbf{U}^0$ which is specified using the surface $\Pi$. Suppose $U^0$ is a certain single-valued sample of the multivalued function $\mathbf{U}^0$. From what has been said above, we obtain that the action of the strategy $U^0$, which is performed with errors in the set $\Pi^r$, is also estimated by the right-hand side of relation (1.3). It is therefore possible to speak of the stability of the strategy $U^0$ with respect to inaccuracies in the construction of the surface $\Pi$.

Assuming that the approximate game is identical with the initial game and that $B^{(3)} = B^{(1)}$, then

$$\chi(t_0, \vartheta) = 0, \quad \|\gamma^{(1)} - \gamma^{(2)}\|_F = 0$$

Furthermore, suppose the value function $\Gamma^{(1)}$ of the initial game is used as the $u$-stable function $V^{(2)}$ and that Condition A is satisfied. By virtue of estimate (1.3), we obtain

$$\Gamma^{(1)}(t_0, x_0, U^0, \Delta) \le \Gamma^{(1)}(t_0, x_0) + 2\lambda\sqrt{(2\sigma\mu\Delta + r)\beta\mu}(\vartheta - t_0) + 4\lambda\sigma\mu\Delta + \lambda r$$

Consequently, if the function $B^{(1)}$ and, also, the pay function $\gamma^{(1)}$ satisfy the Lipschitz condition $\gamma^{(1)}(x) \to \infty$ as $|x| \to \infty$, and if Condition A is satisfied for the value function $\Gamma^{(1)}$ in the pair with the function $B^{(1)}$ and the switching surface $\Pi$ is constructed on the basis of the function $\Gamma^{(1)}$, then the strategy $U^0$ can be taken as the universal, stable, optimal strategy $U^*$ in the game (1.1).

## 2. AUXILIARY ASSERTIONS

Suppose

$$d(X, Y) = \max_{x \in X} \min_{y \in Y} |x - y|$$

is the Hausdorff divergence of the set $X$ from the set $Y$ for compact sets $X$ and $Y$ in $R^n$. We put

$$G_\upsilon^{(i)}(t_*, t^*) = \bigcup_{\upsilon(\cdot) \in L^{(i)}} \int_{t_*}^{t^*} C^{(i)}(t)\upsilon(t)dt, \quad i = 1, 2$$

The sets $G_\upsilon^{(i)}(t_*, t^*)$ are convex compacta. The limit

$$d(G_\upsilon^{(1)}(t_*, t^*), G_\upsilon^{(2)}(t_*, t^*)) \le \int_{t_*}^{t^*} m(t)dt \tag{2.1}$$

holds.

The permissibility set of system (1.2) at the instant of time $t$ for an initial state $x_*$ at the instant of time $t_*$ and in the case of an exhaustive search for all of the permissible preset controls $u(\cdot)$, $\upsilon(\cdot)$ in the interval $[t_*, t]$ is denoted by the symbol $G^{(2)}(t; t_*, x_*)$. We put

$$\mathbf{G}^{(2)}(t; t_*, x_*) = G^{(2)}(t; t_*, x_*) + B(2(t - t_*)\sigma\mu)$$

Here, $B(r)$ is a sphere of radius $r$ in $R^n$.

For $t \in T$ and $c \in R$, we put

$$W_c^{(2)}(t) = \{x \in R^n : V^{(2)}(t, x) \le c\}, \quad W_c^{(2)} = \{(t, x) \in Z : x \in W_c^{(2)}(t)\}$$

*Lemma* 1. Suppose $(t_*, x_*) \in Z$, $\delta > 0$, $t_* + \delta \le \vartheta$ and that $y^{(1^*)}(\cdot)$ is the motion of system (1.1), by virtue of the permissible preset controls $u(\cdot)$, $\upsilon(\cdot)$, which, at the instant of time $t_*$, emerges from the point $x_*$. The estimate

$$\mathcal{V}(t_* + \delta, y^{(1^*)}(t_* + \delta)) \le V^{(2)}(t_*, x_*) + \lambda\beta\mu\delta^2 + \lambda\chi(t_*, t_* + \delta) \tag{2.2}$$

then holds.

*Proof.* Using a control $\upsilon(\cdot) \in L^{(1)}$ specified in the condition of the lemma, we define the point

$$g = \int_{t_*}^{t_* + \delta} C^{(1)}(t)\upsilon(t)dt$$

in the set $G_\upsilon^{(1)}(t_*, t_* + \delta)$.

Suppose $\bar{g}$ is the closest point of the set $G_\upsilon^{(2)}(t_*, t_* + \delta)$ to it. We choose $\bar{\upsilon}(\cdot) \in L^{(2)}$ such that

$$\bar{g} = \int_{t_*}^{t_* + \delta} C^{(2)}(t)\bar{\upsilon}(t)dt$$

Using the $u$-stability of the function $V^{(2)}$, we obtain a $\bar{\upsilon}(\cdot)$ along the direction $\bar{u}(\cdot)$ such that the inclusion

$$y^{(2^*)}(t_* + \delta) \in W_{c_*}^{(2)}(t_* + \delta) \left(c_* = V^{(2)}(t_*, x_*)\right) \tag{2.3}$$

is satisfied in the case of the motion $y^{(2^*)}(t) = y^{(2^*)}(t; t_*, x_*, \bar{u}(\cdot), \bar{\upsilon}(\cdot))$ which emerges from the point $x_*$ at the instant of time $t_*$.

We now use the notation

$$J_1 = \int_{t_*}^{t_*+\delta} B^{(1)}(t)u(t)dt - \int_{t_*}^{t_*+\delta} B^{(2)}(t)\bar{u}(t)dt$$

$$J_2 = \int_{t_*}^{t_*+\delta} C^{(1)}(t)\upsilon(t)dt - \int_{t_*}^{t_*+\delta} C^{(2)}(t)\bar{\upsilon}(t)dt$$

Then

$$y^{(1^*)}(t_* + \delta) - y^{(2^*)}(t_* + \delta) = J_1 + J_2 \tag{2.4}$$

We have

$$\begin{aligned} J_1 = {} & \int_{t_*}^{t_*+\delta} (B^{(1)}(t) - B^{(3)}(t))u(t)dt - \int_{t_*}^{t_*+\delta} (B^{(2)}(t) - B^{(3)}(t))\bar{u}(t)dt + \\ & + \int_{t_*}^{t_*+\delta} \left(B^{(3)}(t) - B^{(3)}(t_* + \delta)\right)(u(t) - \bar{u}(t))dt + B^{(3)}(t_* + \delta) \int_{t_*}^{t_*+\delta} (u(t) - \bar{u}(t))dt \end{aligned} \tag{2.5}$$

$$J_2 = g - \bar{g} \tag{2.6}$$

We denote the operator for the orthogonal projection of the space $R^n$ onto the subspace, which is orthogonal to the vector $B^{(3)}(t_* + \delta)$, by the symbol $\pi$.

Bearing in mind the fact that the controls $u(t)$ and $\bar{u}(t)$ are bounded in modulus by the number $\mu$, the function $B^{(3)}$ satisfies the Lipschitz condition with a constant $\beta$ and that $\pi B^{(3)}(t_* + \delta) = 0$, from relation (2.5) we obtain

$$|\pi J_1| \le \mu \int_{t_*}^{t_*+\delta} \kappa(t)dt + \beta\mu\delta^2$$

Taking into account relations (2.6) and (2.1), we have

$$|\pi J_2| = |\pi g - \pi\bar{g}| \le |g - \bar{g}| \le \int_{t_*}^{t_*+\delta} m(t)dt$$

Finally, we obtain

$$|\pi y^{(1^*)}(t_* + \delta) - \pi y^{(2^*)}(t_* + \delta)| \le \beta\mu\delta^2 + \chi(t_*, t_* + \delta) \tag{2.7}$$

Suppose $\tilde{x}$ is the point on the line $\mathcal{A}(t_* + \delta, y^{(1^*)}(t_* + \delta))$ which is closest to the set $W^{(2)}_{c_*}(t_* + \delta)$. It follows from inclusion (2.3) and the definition of the operator $\pi$ that

$$d(\{\tilde{x}\}, W^{(2)}_{c_*}(t_* + \delta)) \le |\pi\tilde{x} - \pi y^{(2^*)}(t_* + \delta)| = |\pi y^{(1^*)}(t_* + \delta) - \pi y^{(2^*)}(t_* + \delta)|$$

Hence

$$\begin{aligned} V^{(2)}(t_* + \delta, \tilde{x}) &\le c_* + \lambda|\pi y^{(1^*)}(t_* + \delta) - \pi y^{(2^*)}(t_* + \delta)| = \\ &= V^{(2)}(t_*, x_*) + \lambda|\pi y^{(1^*)}(t_* + \delta) - \pi y^{(2^*)}(t_* + \delta)| \end{aligned}$$

Taking inequality (2.7) into account, we conclude that the required inequality follows from the fact that

$$\mathcal{V}(t_* + \delta, y^{(1^*)}(t_* + \delta)) \le V^{(2)}(t_* + \delta, \tilde{x})$$

*Lemma* 2. Suppose $(t_*, x_*) \in Z$, $\delta > 0$, $t_* + \delta \le \vartheta$ and that $y^{(1^*)}(\cdot)$ is the motion of system (1.1) by virtue of the constant control and a certain $\upsilon(\cdot) \in L^{(1)}$ which emerges from the point $x_*$ at the instant of time $t_*$. We assume that

$$\mathbf{G}^{(2)}(t_* + \delta;\, t_*, x_*) \subset \Pi_+(t_* + \delta)\ (\mathbf{G}^{(2)}(t_* + \delta;\, t_*, x_*) \subset \Pi_-(t_* + \delta))$$

The estimate

$$V^{(2)}(t_* + \delta, y^{(1^*)}(t_* + \delta)) \le V^{(2)}(t_*, x_*) + \lambda\beta\mu\delta^2 + \lambda\chi(t_*, t_* + \delta) \tag{2.8}$$

then holds.

*Proof.* As in the initial part of the proof of Lemma 1, we choose a control $\bar{\upsilon}(\cdot) \in L^{(2)}$ using the specified $\upsilon(\cdot) \in L^{(1)}$. Next, using the $u$-stability of the function $V^{(2)}$, we select $\bar{u}(\cdot)$ such that the motion $y^{(2^*)}(\cdot)$, which arises by virtue of $\bar{u}(\cdot)$, $\bar{\upsilon}(\cdot)$, satisfies the conditions

$$y^{(2^*)}(t_*) = x_*, \quad y^{(2^*)}(t_* + \delta) \in W^{(2)}_{c_*}(t_* + \delta) \quad (c_* = V^{(2)}(t_*, x_*)) \tag{2.9}$$

We put

$$\hat{z} = y^{(2^*)}(t_* + \delta) + B^{(3)}(t_* + \delta) \int_{t_*}^{t_* + \delta} (u(t) - \bar{u}(t))dt$$

We will show that

$$V^{(2)}(t_* + \delta, \hat{z}) \le V^{(2)}(t_* + \delta, y^{(2^*)}(t_* + \delta)) \tag{2.10}$$

Consider the case

$$u(t) \equiv \mu, \quad \mathbf{G}^{(2)}(t_* + \delta;\, t_*, x_*) \subset \Pi_+(t_* + \delta)$$

By virtue of the last imbedding, we obtain

$$y^{(2^*)}(t_* + \delta) \in \Pi_+(t_* + \delta), \quad \hat{z} \in \Pi_+(t_* + \delta) \tag{2.11}$$

Since $u(t) \ge \bar{u}(t)$, $t \in [t_*, t_* + \delta]$, the vectors $\hat{z} - y^{(2^*)}(t_* + \delta)$ and $B^{(3)}(t_* + \delta)$ are codirected. On taking Condition A into account, we derive inequality (2.10) from this.

In the case when

$$u(t) \equiv -\mu, \quad \mathbf{G}^{(2)}(t_* + \delta;\, t_*, x_*) \subset \Pi_-(t_* + \delta)$$

inequality (2.10) is proved in a similar manner, only now it is necessary to use relations which differ form (2.11) by the replacement of the plus sign by a minus sign, and the inequality $u(t) \le \bar{u}(t)$, $t \in [t_*, t_* + \delta]$.

Since the right-hand side of inequality (2.10) does not exceed $c_*$, we obtain the inclusion $\hat{z} \in W^{(2)}_{c_*}(t_* + \delta)$. Therefore

$$d(\{y^{(1^*)}(t_* + \delta)\}, W^{(2)}_{c_*}(t_* + \delta)) \le |y^{(1^*)}(t_* + \delta) - \hat{z}|$$

Using the definition of the vector $\hat{z}$ in equality (2.4), we have

$$y^{(1^*)}(t_* + \delta) - \hat{z} = J_1 + J_2 - B^{(3)}(t_* + \delta) \int_{t_*} (u(t) - \bar{u}(t))dt$$

Taking into account equalities (2.5) and (2.6), the Lipschitz condition for the function $B^{(3)}$, the rule for selecting the control $\bar{\upsilon}(\cdot)$ and inequality (2.1), we obtain

$$|y^{(1^*)}(t_* + \delta) - \hat{z}| \le \beta\mu\delta^2 + \chi(t_*, t_* + \delta)$$

The required inequality (2.8) follows from the fact that

$$V^{(2)}(t_* + \delta, y^{(1^*)}(t_* + \delta)) \le V^{(2)}(t_* + \delta, \hat{z}) + \lambda|y^{(1^*)}(t_* + \delta) - \hat{z}|$$

$$V^{(2)}(t_* + \delta, \hat{z}) \le V^{(2)}(t_*, x_*)$$

*Lemma* 3. Suppose $(\bar{t}, \bar{x}) \in Z$, $\hat{t} \in (\bar{t}, \vartheta]$ and that $y^{(1^*)}(\cdot)$ is the motion of system (1.1) by virtue of a constant control $u(t) \equiv \mu$ $(u(t) \equiv -\mu)$ and a certain $\upsilon(\cdot) \in L^{(1)}$ which emerges from the point $\bar{x}$ at a certain instant of time $\bar{t}$. We assume that $y^{(1^*)}(t) \in \Pi_+(t)$ $(y^{(1^*)}(t) \in \Pi_-(t))$ for all $t \in [\bar{t}, \hat{t}]$.

The limit

$$V^{(2)}(\hat{t}, y^{(1^*)}(\hat{t})) \le V^{(2)}(\bar{t}, \bar{x}) + \lambda\chi(\bar{t}, \hat{t}) \tag{2.12}$$

then holds.

*Proof.* We divide the interval $[\bar{t}, \hat{t}]$ into instants $t_1, t_2, \dots, t_s$, $(t_1 = \bar{t}, t_s = \hat{t})$ with a step $\delta$, such that the relation

$$\mathbf{G}^{(2)}(t_{k+1}, t_k, y^{(1^*)}(t_k)) \subset \Pi_+(t_{k+1}) \; (\mathbf{G}^{(2)}(t_{k+1}, t_k, y^{(1^*)}(t_k)) \subset \Pi_-(t_{k+1}))$$

is satisfied for any $k = 1, 2, \dots, s-1$. This can be done on the basis of the assumption concerning the location of $y^{(1^*)}(t)$ relative to $\Pi(t)$. By virtue of Lemma 2, we have the estimate

$$V^{(2)}(t_{k+1}, y^{(1^*)}(t_{k+1})) \le V^{(2)}(t_k, y^{(1^*)}(t_k)) + \lambda\beta\mu\delta^2 + \lambda\chi(t_k, t_{k+1})$$

On applying it successively for $k = 1, 2, \dots, s-1$, we prove an equality which differs form (2.2) in that there is a term $\lambda\beta\mu\delta(\hat{t}, \bar{t})$ on the right-hand side. On taking the limit when $\delta \to 0$, we obtain the limit (2.12).

*Lemma* 4. Suppose $(\bar{t}, \bar{x}) \in Z$, $\hat{t} \in (\bar{t}, \vartheta]$ and that $y^{(1^*)}(\cdot)$ is the motion of system (1.1), by virtue of the permissible preset controls $u(\cdot)$, $\upsilon(\cdot)$, which emerges from the point $\bar{x}$ at the instant of time $\bar{t}$. The limit

$$V^{(2)}(\hat{t}, y^{(1^*)}(\hat{t})) \le V^{(2)}(\bar{t}, \bar{x}) + 2\lambda\mu\sigma(\hat{t} - \bar{t}) + \lambda\chi(\bar{t}, \hat{t}) \tag{2.13}$$

then holds.

*Proof.* We assume that $(t_*, x_*) \in Z$, $\delta > 0$, $t_* + \delta \le \vartheta$. Copying the initial part of the proof of Lemma 1, using the specified $\upsilon(\cdot) \in L^{(1)}$, we select the extremal control $\bar{\upsilon}(\cdot) \in L^{(2)}$. We then select $\bar{u}(\cdot)$ such that the motion $y^{(2^*)}(\cdot)$, which arises by virtue of $\bar{u}(\cdot)$, $\bar{\upsilon}(\cdot)$, satisfies conditions (2.9).

On taking account of equalities (2.4)–(2.6), the Lipschitz conditions for the function $B^{(3)}$, the inequality $|B^{(3)}(t_* + \delta)| \le \sigma$, the rule for selecting the control $\bar{\upsilon}(\cdot)$ and inequality (2.1), we obtain

$$|y^{(1^*)}(t_* + \delta) - y^{(2^*)}(t_* + \delta)| \le \beta\mu\delta^2 + 2\sigma\mu\delta + \chi(t_*, t_* + \delta)$$

By virtue of the relations

$$V^{(2)}(t_* + \delta, y^{(1^*)}(t_* + \delta)) \le V^{(2)}(t_* + \delta, y^{(2^*)}(t_* + \delta)) + \lambda|y^{(1^*)}(t_* + \delta) - y^{(2^*)}(t_* + \delta)|$$

$$V^{(2)}(t_* + \delta, y^{(2^*)}(t_* + \delta)) \le V^{(2)}(t_*, x_*)$$

we derive the inequality

$$V^{(2)}(t_* + \delta, y^{(1^*)}(t_* + \delta)) \le V^{(2)}(t_*, x_*) + \lambda\beta\mu\delta^2 + 2\lambda\sigma\mu\delta + \lambda\chi(t_*, t_* + \delta) \tag{2.14}$$

from this.

As in the proof of Lemma 3, subdividing the interval $[\bar{t}, \hat{t}]$ with a step $\delta$, we use the estimate (2.14) at each step and, taking the limit as $\delta \to 0$, we obtain the limit (2.13).

## 3. PROOF OF THE THEOREM

We fix a number $r \ge 0$ and consider the motion $y^{(1)}(\cdot)$ of system (1.1) from a position $(t_0, x_0) \in K$, $t_0 < \vartheta$ by virtue of a certain strategy $U \subset \mathbf{U}$ of the first player with a step $\Delta$ of the discrete control system and a certain $\upsilon(\cdot) \in L^{(1)}$.

In order to describe the change in the function $V^{(2)}$ along the motion $y^{(1)}(\cdot)$ in the interval $[t_*, t^*]$, we introduce the notation

$$\operatorname{Var}(V^{(2)}, [t_*, t^*]) = V^{(2)}(t^*, y^{(1)}(t^*)) - V^{(2)}(t_*, y^{(1)}(t_*))$$

1. Suppose $\beta > 0$, $\sigma > 0$. We put

$$h = \sqrt{(2\sigma\mu\Delta + r)/(\beta\mu)} \tag{3.1}$$

(A) Along the motion $y^{(1)}(\cdot)$, we separate out the "loops" which are associated with entry into the sets $\Pi^r(t)$. We also determine the free intervals.

On moving from $t_0$ to $\vartheta$, we find the first instant of time $t$, when $y^{(1)}(t) \in \Pi^r(t)$. We call this instant the instant of the start of the first loop and we denote it by $t_1$. Next, we note the instant $\tilde{t}_1$ of the termination of the first loop as the last instant $t$ in the interval $[t_1, t_1 + h] \cap T$ at which $y^{(1)}(t) \in \Pi^r(t)$. The instant $\tilde{t}_1$, in particular, can be identical to $t_1$.

We take the first instant $t \in [t_1 + h, \vartheta]$, when $y^{(1)}(t) \in \Pi^r(t)$ as the instant, $t_2$, of the start of the second loop. We then note the instant $\tilde{t}_2$ of the termination of the second loop as the last instant $t$ in the interval $[t_2, t_2 + h] \cap T$, when $y^{(1)}(t) \in \Pi^r(t)$.

Continuing this process, we obtain the set of loops in $[t_0, \vartheta]$.

We now remove the domain of the intervals of the loops which have been constructed from $[t_0, \vartheta]$ and we obtain an ordered set of segments. We call each of them a free interval which may be degenerate, that is, consist of a single point.

If there are no loops in $[t_0, \vartheta]$, then we assume that $[t_0, \vartheta]$ is the free interval.

(B) Suppose $[\tau, \eta]$ is a certain free interval. We will show that an increment of the function $V^{(2)}$ in it is described by the inequality

$$\mathrm{Var}_f(V^{(2)}, [\tau, \eta]) \le 2\lambda\sigma\mu\Delta + \lambda\chi(\tau, \eta) \tag{3.2}$$

The subscript $f$ emphasizes that the change in the function $V^{(2)}$ is calculated in the free interval.

A certain control $u(\cdot)$ is realized along the motion $y^{(1)}(\cdot)$. We call the value $u(t)$ a "correct" value if $u(t) = \mu$ ($u(t) = -\mu$) when $y^{(1)}(t) \in \Pi_+(t)$ ($y^{(1)}(t) \in \Pi_-(t)$).

In the domain of the free interval, the motion $y^{(1)}(\cdot)$ goes along one side of the set $\Pi^r$ and, therefore, along one side of the surface $\Pi$. Hence, when $\Delta \le \eta - \tau$, the control $u(t)$ is correct in $[\tau + \Delta, \eta]$ and arbitrary, perhaps, only in $[\tau, \tau + \Delta]$. By virtue of Lemma 3, we obtain

$$\mathrm{Var}(V^{(2)}, [\tau + \Delta, \eta]) \le \lambda\chi(\tau + \Delta, \eta)$$

and, by virtue of Lemma 4,

$$\mathrm{Var}(V^{(2)}, [\tau, \tau + \Delta]) < 2\lambda\mu\sigma\Delta + \lambda\chi(\tau, \tau + \Delta)$$

On summing the last two inequalities, we arrive at the limit (3.2).

If $\Delta > \eta - \tau$, we apply Lemma 4 to the whole of the interval $[\tau, \eta]$. We again obtain the limit (3.2).

(C) We shall say that $[\tau, \eta]$ is an interval of the form $E_1$ if it consists of a certain loop $[t_i, \tilde{t}_i]$ and a free interval adjacent to the right of it. We shall call an interval $[\tau, \eta]$ of the form $E_1$ with the additional condition $\tau + h \le \eta$ an interval of the form $E_2$.

We will now evaluate the increment of the function $V^{(2)}$ along the motion $y^{(1)}(\cdot)$ in an interval of the form $E_1$.

We consider the interval of the loop $[t_i, \tilde{t}_i]$. Applying Lemma 1 when $\delta = \tilde{t}_i - t_i$, we have

$$\mathcal{V}(\tilde{t}_i, y^{(1)}(\tilde{t}_i)) \le V^{(2)}(t_i, y^{(1)}(t_i)) + \lambda\beta\mu(\tilde{t}_i - t_i)^2 + \lambda\chi(t_i, \tilde{t}_i)$$

Since $\tilde{t}_i - t_i \le h$, the second term on the right-hand side can be replaced by $\lambda\beta\mu h(\tilde{t}_i - t_i)$.

On taking account of the inequality

$$V^{(2)}(\tilde{t}_i, y^{(1)}(\tilde{t}_i)) \le \mathcal{V}(\tilde{t}_i, y^{(1)}(\tilde{t}_i)) + \lambda r$$

we arrive at the relation

$$\mathrm{Var}(V^{(2)}, [t_i, \tilde{t}_i]) \le \lambda\beta\mu h(\tilde{t}_i - t_i) + \lambda r + \lambda\chi(t_i, \tilde{t}_i) \tag{3.3}$$

In the free interval $[\bar{t}_i, \eta]$, we have inequality (3.2) when $\tau = \bar{t}_i$ and, when this is combined with inequality (3.3), taking account of the inequality $\bar{t}_i - t_i \le \eta - \tau$, we obtain

$$\mathrm{Var}_1(V^{(2)}, [\tau, \eta]) \le \lambda\beta\mu h(\eta - \tau) + 2\lambda\sigma\mu\Delta + \lambda r + \lambda\chi(\tau, \eta) \tag{3.4}$$

The subscript 1 emphasizes that the calculation of the increment in the function $V^{(2)}$ takes place in an interval of the form $E_1$.

We will now evaluate the increment $\mathrm{Var}_2$ of the function $V^{(2)}$ along the motion $y^{(1)}(\cdot)$ in an interval of the form $E_2$. Since, $\eta - \tau \ge h$ in this case, the inequality

$$2\lambda\sigma\mu\Delta + \lambda r \le \lambda\beta\mu h(\eta - \tau)$$

follows from relation (3.1).

On invoking inequality (3.4), we obtain

$$\mathrm{Var}_2(V^{(2)}, [\tau, \eta]) \le 2\lambda\beta\mu h(\eta - \tau) + \lambda\chi(\tau, \eta) \tag{3.5}$$

(D) We will now consider the interval $[t_0, \vartheta]$ and represent it as being composed of the first free interval $[t_0, t_1]$, a finite number of intervals of the form $E_2$, which go one after the other from the instant $t_1$ to a certain instant $t^*$ (their total interval is $[t_1, t^*]$), and the remaining interval $[t^*, \vartheta]$ of the form $E_1$. On successively applying limits (3.2), (3.5) and (3.4), we have

$$\begin{aligned}\mathrm{Var}(V^{(2)}, [t_0, \vartheta]) &= \mathrm{Var}_f(V^{(2)}, [t_0, t_1]) + \mathrm{Var}(V^{(2)}, [t_1, t^*]) + \\ &+ \mathrm{Var}_1(V^{(2)}, [t^*, \vartheta]) \le 2\lambda\sigma\mu\Delta + 2\lambda\beta\mu h(t^* - t_1) + \\ &+ \lambda\beta\mu h(\vartheta - t^*) + 2\lambda\sigma\mu\Delta + \lambda r + \lambda\chi(t_0, \vartheta) \le \\ &\le 2\lambda\beta\mu h(\vartheta - t_0) + 4\lambda\sigma\mu\Delta + \lambda r + \lambda\chi(t_0, \vartheta)\end{aligned}$$

Substituting $h$ using formula (3.1), we obtain

$$\mathrm{Var}(V^{(2)}, [t_0, \vartheta]) \le \Lambda(t_0, r, \Delta) \tag{3.6}$$

2. Suppose $\beta = 0$, $\sigma \ge 0$. On moving from $t_0$ to $\vartheta$, we find the first instant $t$ when $y^{(1)}(t) \in \Pi^r(t)$. This is denoted by $t_1$. Suppose $\hat{t}$ is the last instant in $[t_0, \vartheta]$ when $y^{(1)}(t) \in \Pi^r(t)$.

We have

$$y^{(1)}(t) \notin \Pi^r(t), \quad t \in [t_0, t_1) \cup (\hat{t}, \vartheta]$$

On the basis of Lemma 3 and 4 (as when deriving inequality (3.2)), we obtain

$$\mathrm{Var}(V^{(2)}, [t_0, t_1]) \le 2\lambda\sigma\mu\Delta + \lambda\chi(t_0, t_1) \tag{3.7}$$

$$\mathrm{Var}(V^{(2)}, [\hat{t}, \vartheta]) \le 2\lambda\sigma\mu\Delta + \lambda\chi(\hat{t}, \vartheta) \tag{3.8}$$

for the intervals $[t_0, t_1]$ and $[\hat{t}, \vartheta]$.

For the interval $[t_1, \hat{t}]$, using Lemma 1 with $\beta = 0$, we have

$$\mathcal{V}(\hat{t}, y^{(1)}(\hat{t})) \le V^{(2)}(t_1, y^{(1)}(t_1)) + \lambda\chi(t_1, \hat{t})$$

and, hence, on taking account of the inequality

$$V^{(2)}(\hat{t}, y^{(1)}(\hat{t})) \le \mathcal{V}(\hat{t}, y^{(1)}(\hat{t})) + \lambda r$$

we arrive at the limit

$$\mathrm{Var}(V^{(2)}, [t_1, \hat{t}]) \le \lambda r + \lambda\chi(t_1, \hat{t}) \tag{3.9}$$

Combining inequalities (3.7)–(3.9), we obtain

$$\mathrm{Var}(V^{(2)}, [t_0, \vartheta]) \le 4\lambda\sigma\mu\Delta + \lambda r + \lambda\chi(t_0, \vartheta) \tag{3.10}$$

3. Using inequality (3.6) when $\beta > 0$, $\sigma > 0$ and inequality (3.10) when $\beta = 0$, $\sigma \ge 0$, we have the limit

$$V^{(2)}(\vartheta, y^{(1)}(\vartheta)) \le V^{(2)}(t_0, x_0) + \Lambda(t_0, r, \Delta) \tag{3.11}$$

Since

$$\gamma^{(2)}(y^{(1)}(\vartheta)) = V^{(2)}(\vartheta, y^{(1)}(\vartheta)), \quad \gamma^{(1)}(y^{(1)}(\vartheta)) \le \gamma^{2}(y^{(1)}(\vartheta)) + \|\gamma^{(1)} - \gamma^{(2)}\|_F$$

and the right-hand side of inequality (3.11) is independent of the chosen $\upsilon(\cdot) \in L^{(1)}$, we conclude that inequality (1.3) holds.

## 4. TEST OF THE NUMERICAL CONSTRUCTION OF THE SWITCHING SURFACE

Algorithms for the numerical construction of switching surfaces in linear differential games with a fixed instant of termination are not discussed in this paper. We will confine ourselves to a brief description of publications where the results of computer modelling using switching surfaces have been reported.

The simplest case is when, in a linear differential game, the values of the quasiconvex pay function at the instant when the game terminates are determined by only two certain coordinates of the phase vector, that is, $n = 2$.

Efficient algorithms for constructing $t$-sections of sets of the level of the value function in the coordinates of system (1.1) have been developed in [10–12] for this case. Discretization with respect to $t$ determines the approximating game (1.2). The constructions are carried out in a specified mesh $\{t_k\}$ of the instant of time and in a certain mesh $\{c_p\}$ of the values of the value function. Each section $W_c^{(2)}(t_k)$ of the set for a level is a convex polygon in a plane. The transition from the section which has been constructed $W_c^{(2)}(t_k)$ to the section $W_c^{(2)}(t_{k-1})$, $t_{k-1} < t_k$ is accomplished using a retrograde procedure which uses the operation of making a positive-homogeneous, piecewise-linear function convex in the space $R^n$.

Simple processing [5, 7, 11, 12] of the polygons $W_c^{(2)}(t_k)$, $c \in \{c_p\}$ gives the switching line, corresponding to the instant $t_k$, for the control action $u$ of the first player. The switching lines which have been calculated in the mesh $\{t_k\}$ give the switching lines in the space of the game. The sets of switching lines are stored in a memory and are used in the discrete control scheme.

The problem of the landing of an aircraft under conditions of windshear has been investigated [11, 13–16] using the above-mentioned programs. The landing process was considered up to the instant when the end of the runway was reached. A control procedure using a switching surface, which is specified by a set of switching lines, was also tested [17, 18] on model landing and take-off problems from [9–21]. A problem on the take-off run of an aircraft along the runway under conditions of windshear was considered in [22]; the control procedure investigated was also based on the construction of a switching surface.

The problem of the shifting of a load from a fixed point of suspension has been studied in a game formulation in [23]; a switching surface was constructed which determines the optimal control procedure.

A software package for constructing switching surfaces in the case when $n = 3$ has been described in [24].

I wish to thank L. V. Kamneva for useful remarks.

This research was supported by the Russian Foundation for Basic Research (03-01-00415).

## REFERENCES

1. KRASOVSKII, N. N. and SUBBOTIN, A. I., *Positional Differential Games*. Nauka, Moscow, 1974.
2. KRASOVSKII, N. N. and SUBBOTIN, A. I., *Game-Theoretical Control Problems*. Springer, New York, 1988. 518.
3. KRASOVSKII, N. N., Differential games. Approximate and formal models. *Mat. Sbomik*, 1978, **107**, 4, 541–571.
4. SUBBOTINA, N. N., Universal optimal strategies in positional differential games, *Differents. Uravneniya*, 1983, **19**, 11, 1890–1896.
5. BOTKIN, N. D. and PATSKO, V. S., Universal strategy in a differential game with fixed terminal time. *Problems of Control and Inform. Theory*, 1982, **11**, 419–432.
6. BOTKIN, N. D., Optimal universal strategy in a linear differential game. *Differents. Uravneniya*, 1989, **25**, 9, 1475–1480.
7. BOTKIN, N. D. and PATSKO, V. S., Position control in a linear differential game. *zv. Akad. Nauk SSSR. Tekhn. Kibernetika*, 1983, 4, 78–85.
8. ZARCH, M. A. and PATSKO, V. S., Construction of a second player control in a linear differential game on the basis of a repulsion property. In *Controls with a Guaranteed Result*. (Edited by A. I. Subbotin and V. N. Ushakov). Inst. Mat. Mekh. Ural'sk. Nauch. Tsentra Akad. Nauk SSSR, Sverdlovsk, 1984, 395–400.

9. ZARCH, M. A., Universal strategy of the second player in a linear differential game, *Prikl. Mat. Mekh.*, 1990, **54**, 3, 395–400.
10. ISAKOVA, Ye. A., LOGUNOVA, G. V. and PATSKO, V. S., Construction of stable bridges in a linear differential game with a fixed instant of termination. In Algorithms and Programs for Solving Linear Differential Games (Edited by A. I. Subbotin and V. S. Patsko). Inst. Mat. Mekh. Ural'sk. Nauch, Tsentra Akad. Nauk SSSR, Sverdlovsk, 1984, 127–158.
11. BOTKIN, N. D., PATSKO, V. S. and ZARKH, M. A., Numerical solution of linear differential game. *Differential Games – Developments in Modelling and Computation: Lecture Notes in Control and Inform. Sci.*, Springer, Berlin, 1991, **156**, 226–234.
12. PATSKO, V. S., Special aspects of convex hull constructing in linear differential games of small dimension. *Proc. 10th IFAC Workshop on Control Applications of Optimization*. Pergamon Press. London, 1996. 19–24.
13. BOTKIN, N. D., KEIN, V. M. and PATSKOV, V. S., A model problem on the control of the lateral motion of an aircraft on landing. *Prikl. Mat. Mekh.*, 1984, **48**, 4, 560–567.
14. BOTKIN, N. D., KEIN, V. M., PATSKO, V. S. and TUROVA, V. L., Aircraft landing control in the presence of windshear. *Problems of Control and Inform. Theory*, 1989, **18**, 223–235.
15. BOTKIN, N. D., ZARCH, M. A., KEIN, V. M., PATSKO, V. S. and TUROVA, V. L., Differential games and problems of controlling an aircraft in the case of windshear. *Izv. Ross. Akad. Nauk. Tekhn. Kibernetika*, 1993, 1, 68–76.
16. PATSKO, V. S., BOTKIN, N. D., KEIN, V. M., TUROVA, V. L. and ZARK, M. A., Control of an aircraft landing in windshear. *J. Optimiz. Theory Appl.*, 1994, **83**, 237–267.
17. IVANOV, A. G., Modelling of the motion of an aircraft during the landing stage. In *Control Problems with a Guaranteed Result* (Edited by A. I. Subbotin and S. A. Brykalov). Inst. Mat. Ural'sk. Mekh., Otd. Ross. Akad. Nauk, Ekaterinburg, 1992, 15–26.
18. TUROVA, V. L., Application of numerical methods in the differential game theory to problems on the take-off and the termination of landing of an aircraft. *Trudy Inst. Mat. Mekh. Ural'sk. Otd. Ross. Akad. Nauk*, 1984, **2**, 188–201.
19. MIELE, A., WANG, T. and MELVIN, W. W., Optimal take-off trajectories in the presence of windshear. *J. Optimiz. Theory Appl.*, 1986, **49**, 1–45.
20. MIELE, A., WANG, T., TZENG, C. Y. and MELVIN, W. W., Optimal abort landing trajectories in the presence of windshear. *J. Optimiz. Theory Appl.*, 1987, **55**, 165–202.
21. MIELE, A., WANG, T., WANG, H. and MELVIN, W. W., Optimal penetration landing trajectories in the presence of windshear. *J. Optimiz. Theory Appl.*, 1988, **57**, 1–40.
22. BOTKIN, N. D. and KRASOV, A. I., Positional controls in a model problem of the take-off of an aircraft. In Positional Control with a Guaranteed Result (Edited by A. I. Subbotin and A. M. Taras'yeva). Inst. Mat. Mekh. Ural'sk. Otd. Akad. Nauk SSSR, Sverdlovsk, 1988, 22–32.
23. SOKOLOV, B. N. and TUROVA, V. L., Synthesis of the optimal control of a pendulum in the presence of active interference. *Izv. Ross. Akad. Nauk. MTT.*, 1988, 5, 14–23.
24. ZARCH, M. A., A software package for solving three-dimensional differential games. In *Analysis and Synthesis of Dynamical Systems under Conditions of Uncertainty*. Izd. MAI, Moscow, 1990, 35–41.


*Translated by* E.L.S.